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Differential Quadrature Approach for Delamination Buckling Analysis of Composites with Shear Deformation

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The differential quadrature method (DQM) is used to analyze the one-dimensional buckling of a laminated composite beam plate having an across-the-width delamination located at an arbitrary depth and an arbitrary location along its span. A beam theory with shear deformation is used in formulating the problem. Several case studies are conducted to examine the buckling response of laminates hosting such a delamination. Using DQM, the system of equilibrium equations and the boundary conditions are transformed into a system of linear algebraic eigenvalue equations that are solved by a standard eigensolver. The influences of several parameters that affect the buckling strength of such laminates are investigated. The investigated parameters are the shear deformation factor, the length of the delamination, and the through-the-thickness and longitudinal positions of the delamination. The results verify the accuracy and efficiency of DQM.

Introduction

SOLUTIONS of the linear and nonlinear differential equations have always been considered as one of the most significant tasks in science and engineering. Because of the limitation of the analytical solutions, attention has been focused on the development of approximate and numerical methods. The three most popular numerical techniques in use for solving partial differential equations are the finite difference method (FD), the finite element method (FEM), and the boundary element method (BEM). FD is one of the simplest numerical methods (in terms of both its formulation and programming effort). To obtain an accurate result, however, considerable effort is required for representing (discretizing) the domain by a large number of grid points. FEM and BEM on the other hand require more skill and effort for algorithm development and implementation. Thus, the development of new methods from the standpoint of numerical accuracy, ease of formulation, and computational efficiency is still of prime interest.

A relatively new numerical technique is the differential quadrature method (DQM). Bellman and Casti[1] introduced DQM in the early 1970s for solving linear and nonlinear partial differential equations. DQM also has been shown to perform extremely well in solving initial and boundary value problems. There are also numerous applications of DQM in structural mechanics.[2]

This paper presents a new application of DQM for solving the delamination-buckling problem in composite laminates.

Background and Motivation

Delamination buckling of composites is one of the most important modes of failure and governs the compressive response of such materials. A delamination may be initiated during manufacturing of composites or due to impact of an external object in service. Such delaminations can significantly reduce the buckling strength of composites. The parameters that influence the initiation of the buckling are shape, position, number and length of the delamination, geometrical and natural boundary conditions, material properties, stacking sequence, and fiber orientation. Consequently, the phenomenon has attracted the attention of several authors, and various models and solution methods have been presented. Most of these models are based on classical laminate theory, which overestimates the buckling load.[3,4] Moreover, the transverse shear modulus of composite materials is usually much smaller than their in-plane (Young's) modulus. As a result, the effect of transverse shear stress is quite significant even if the beam's length-to-depth ratio is greater than 10 (the limiting ratio often used for isotropic materials). Therefore, one should account for the shear deformation. Consequently, several authors have proposed various solutions considering these parameters. Some of these solutions are briefly discussed next.

Kardomateas and Schmueser[4] used a perturbation technique to analyze the compressive stability of a one-dimensional through-the-width delaminated orthotropic homogeneous elastic beam. They also considered the transverse shear effect on the buckling load and postbuckling behavior of the beam. The classical buckling equations were used, and the effects of transverse shear were accounted for by some correction terms. Using a variational energy approach, Chen[3] formulated the same problem. According to his results, inclusion of the shear deformation causes reduction in the buckling and ultimate strength of delaminated composite plates. Kyoung and Kim[5] used the variational principle to calculate the buckling load and delamination growth of an axially loaded beam plate with an asymmetric delamination (with respect to the center of the plate). They investigated the effects of shear deformation and geometric parameters on the buckling strength and delamination growth of composite beams using their proposed solution and experimental investigation.

The use of various numerical approaches to model the phenomenon has also been reported by several authors. For instance, Lee et al.[6] used an FEM based on a layerwise plate theory to compute the buckling loads and mode shapes of a plate having single and multiple delaminations. Sheinman et al.[7] used FD to solve the differential equations of a composite delaminated beam subjected to arbitrary loading and boundary conditions. Their solution treated the problem in two stages: A nonlinear analysis was used to evaluate the prebuckling response, and an eigenvalue analysis was used to determine the buckling load. An Euler formulation was used to model the buckling response, and bending–stretching coupling was taken into account and was shown to significantly influence the buckling strength. Moradi and Taheri[8] applied DQM to analyze delamination buckling of laminated beam plates using the classical formulation. Their results illustrated the high degree of accuracy that was efficiency obtained by DQM.

In the present work DQM is applied to characterize the buckling response of a unidirectional rectangular beam plate having a through-the-width delamination, as shown in Fig. 1. The delamination can be at any arbitrary location through the thickness and along the span of the plate. The plate can have clamped and/or hinged support conditions at its far ends. The investigation also examines the influence of some of the parameters that can influence the buckling strength of laminates. These parameters include the inclusion of the transverse shear deformation in the formulation and the length and the though-the-thickness position of the delamination. The next section presents the details of the approach, its

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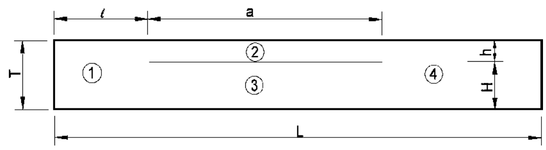

Fig. 1 Model geometry.

formulation, its implementation, and the conversion of the system of differential equations into a system of linear algebraic eigenvalues. The eigenvalues of the system of equations (buckling loads) will be evaluated, and the performances of the method will be discussed through a series of case studies.

DQM

Bellman and Casti[1] originally introduced DQM for solving linear or nonlinear differential equations. It is stated that the partial derivative of a function with respect to a space variable can be approximated by a weighted linear combination of function values at some intermediate points in the domain of that variable. Consider a function $f = f(x)$ on the domain $a \leq x \leq b$. Based on the preceding definition, the nth-order differential of the function f at an intermediate point x_i can be approximated by the weighted linear sum of the function values as

$$\frac{\mathrm{d}^n f(x_i)}{\mathrm{d}x^n} = \sum_{k=1}^{N} c_{ik}^{(n)} f(x_k), \qquad i = 1, \ldots, N \quad n = 1, \ldots, N-1 \tag{1}$$

In Eq. (1), the domains are divided into N discrete points and $c_{ik}^{(n)}$ are the weighting coefficients of the nth derivative, where $n \leq N - 1$. To ensure the integrity of the method, two factors need particular attention. These are 1) the values of the weighting coefficients and 2) the position of the discrete variables. The appropriate values of the weighting coefficients are selected by approximating $f(x)$ by a test function. For instance, one can select a test function in the form of polynomials of order $N - 1$, with the condition that the function would be exact for all polynomials up to $N - 1$ order. However, such functions can produce an ill-conditioned matrix, resulting in inaccuracies of the weightings, as the number of sampling points is increased. Alternatively, the weighting coefficients may be determined explicitly once for all discrete points, irrespective of the position and number of the sampling points, by using the generalized differential quadrature scheme. This approach was proposed independently by Quan and Chang[9] and by Shu and Richards.[10] They used the Lagrange interpolating function as the test function and derived the following recurrence formulas for the weighting coefficients:

$$c_{ij}^{(1)} = \frac{\prod(x_i)}{(x_i - x_j)\cdot\prod(x_j)}, \qquad i, j = 1, \ldots, N \quad \text{and} \quad j \neq i$$

$$c_{ij}^{(k)} = k\left[c_{ii}^{(k-1)}\cdot c_{ij}^{(1)} - \frac{c_{ij}^{(k-1)}}{x_i - x_j}\right], \qquad 2 \leq k \leq N-1 \tag{2}$$

$$c_{ii}^{(m)} = -\sum_{\substack{j=1\\ j\neq i}}^{N} c_{ij}^{(m)}, \qquad m = 1, \ldots, N-1$$

where

$$\prod(x_i) = \prod_{\substack{j=1\\ j\neq i}}^{N} (x_i - x_j) \tag{3}$$

Note that the number of the sampling points used does not affect the performance of the preceding relations; thus, their use can significantly improve the computational performance. Furthermore, in some cases, the spacing of the sampling points can significantly affect the accuracy of the solution. For instance, selection of equally spaced sampling points, a relatively simple and easy mean, often produces less accurate results than unequally spaced sampling points. An effective method is to select the zeros of shifted orthogonal polynomials. A simple and yet effective choice is the roots of shifted Chebyshev polynomials in the [0, 1] domain, presented as

$$x_i = \tfrac{1}{2}\{1 - \cos[(2i-1)/2N]\pi\} \tag{4}$$

Also, the use of zeros of shifted Legendre polynomials has been shown to produce very accurate results.[8,11]

Delamination Buckling

The geometry of the one-dimensional delaminated beam plate used is shown in Fig. 1. This model contains a through-the-width delamination with the length a located arbitrarily through the thickness and along the span of the beam plate. As shown in Fig. 1, the delamination divides the plate into four regions. The part above the delamination plane, with thickness h, is referred to as the upper sublaminate, and the part below it, with thickness H, is referred to as the lower sublaminate. The sections before and after the delamination, where the beam plate is intact, are referred to as the base laminates. To formulate the problem, we consider each of these regions as separate beams. Using the shear deformation theory, the equilibrium equations for each region can be written as

$$P_{i,x} = 0, \qquad i = 1, 2, 3, 4 \tag{5}$$

$$M_{i,x} - Q_i = 0, \qquad i = 1, 2, 3, 4 \tag{6}$$

$$Q_{i,x} - P_i w_{i,xx} = 0, \qquad i = 1, 2, 3, 4 \tag{7}$$

where P_i, Q_i, M_i, and w_i are the in-plane force, shear force, moment, and transverse deflection of each region i, respectively. By substituting

$$M_i = D_i \psi_{i,x} \tag{8}$$

$$Q_i = kGt_i(\psi_i + w_{i,x}) \tag{9}$$

into Eqs. (6) and (7), the governing differential equations reduce to

$$D_i \psi_{i,xx} - kGt_i(\psi_i + w_{i,x}) = 0 \tag{10}$$

$$kGt_i(\psi_i + w_{i,x})_{,x} - P_i w_{i,xx} = 0 \tag{11}$$

where ψ_i and t_i are the transverse rotation angle and thickness of the ith region, respectively; $G = G_{xz}$ is the shear modulus; k is the shear factor; and D_i is the stiffness of the ith part given by

$$D_i = \frac{E_x t_i^3}{12(1 - \nu_{xz}\nu_{zx})} \tag{12}$$

where E_x is Young's modulus along the x direction and ν is Poisson's ratio.

The boundary conditions for the different regions consist of the in-plane, transverse, and continuity conditions. The in-plane and transverse boundary conditions at both ends can be represented by

$$\text{at} \quad x = 0 \rightarrow P_1 = -P, \quad \text{at} \quad x = L \rightarrow P_4 = -P \tag{13}$$

Also, for simply supported ends,

$$\text{at} \quad x = 0 \rightarrow w_1 = \psi_{1,x} = 0, \quad \text{at} \quad x = L \rightarrow w_4 = \psi_{4,x} = 0 \tag{14a}$$

and if the end is clamped, then

$$\text{at} \quad x = 0 \rightarrow w_1 = \psi_1 = 0, \quad \text{at} \quad x = L \rightarrow w_4 = \psi_4 = 0 \tag{14b}$$

Continuity conditions at the delaminated edges consist of the transverse, moment, in-plane, and shear force continuity. At $x = l$,

$$w_1 = w_2 = w_3, \qquad \psi_1 = \psi_2 = \psi_3 \tag{15a}$$

$$D_1\psi_{1,x} - D_2\psi_{2,x} - D_3\psi_{3,x} + (H/2)P_2 - (h/2)P_3 = 0 \tag{15b}$$

$$P_1 - P_2 - P_3 = 0 \tag{15c}$$

$$[kGT(\psi_1 + w_{1,x}) - kGh(\psi_2 + w_{2,x}) - kGH(\psi_3 + w_{3,x})] - (P_1 w_{1,x} - P_2 w_{2,x} - P_3 w_{3,x}) = 0 \tag{15d}$$

At $x = l + a$,

$$w_2 = w_3 = w_4, \qquad \psi_2 = \psi_3 = \psi_4 \tag{15e}$$

$$D_2\psi_{2,x} + D_3\psi_{3,x} - D_4\psi_{4,x} - (H/2)P_2 + (h/2)P_3 = 0 \tag{15f}$$

$$P_2 + P_3 - P_4 = 0 \tag{15g}$$

$$[kGh(\psi_2 + w_{2,x}) + kGH(\psi_3 + w_{3,x}) - kGT(\psi_4 + w_{4,x})]$$
$$- (P_2 w_{2,x} + P_3 w_{3,x} - P_4 w_{4,x}) = 0 \tag{15h}$$

From the primary state solution

$$P_i = (t_i/T)P \tag{16}$$

The boundary conditions include terms containing the in-plane forces, which makes the formation of the problem a matrix form and, hence, the solution of the associated eigenvalues impossible. Therefore, to convert the equilibrium and boundary condition equations into matrix form, the terms associated with the in-plane forces in Eqs. (15b), (15d), (15f), and (15h) must be eliminated. Substituting Eqs. (15a), (15e), and (16) into Eqs. (15d) and (15h) gives

$$Tw_{1,x} - hw_{2,x} - Hw_{3,x} = 0 \tag{17}$$

$$Tw_{4,x} - hw_{2,x} - Hw_{3,x} = 0 \tag{18}$$

which essentially indicates that the first derivatives of the transverse deformations at the delamination fronts are equal for the neighboring regions. Furthermore, applying the axial strain compatibility condition for the upper and lower sublaminates results in

$$\frac{1}{2}\int_0^a \left(\frac{\mathrm{d}w_2}{\mathrm{d}x}\right)^2 \mathrm{d}x + \frac{(1-\nu_{xz}\nu_{zx})P_2 a}{E_x h} = \frac{1}{2}\int_0^a \left(\frac{\mathrm{d}w_3}{\mathrm{d}x}\right)^2 \mathrm{d}x$$
$$+ \frac{(1-\nu_{xz}\nu_{zx})P_3 a}{E_x H} + T\cdot\psi(l) \tag{19}$$

where $\psi(l)$ is the rotation angle at the delamination front. Equation (19) represents the postbuckling behavior of the delaminated beam. Note that, by replacing the ψ term in Eq. (19) with $\frac{1}{2}[\psi(l) + \psi(l+a)]$, the equation will become capable of treating a nonsymmetrical delamination. Before the onset of buckling, the magnitude of the $(\mathrm{d}w_i/\mathrm{d}x)^2$ term is insignificant; hence, Eq. (19) can be reduced to

$$P_2\frac{H}{2} - P_3\frac{h}{2} = \frac{T\cdot h\cdot H\cdot E_x}{2a\cdot(1-\nu_{xz}\nu_{zx})}\cdot\psi(l) \tag{20}$$

Applying Eq. (20) to the moment continuity conditions [Eqs. (15b) and (15f)] at the delamination fronts results in

$$D_1\psi_{1,x} - D_2\psi_{2,x} - D_3\psi_{3,x} + \frac{E_x ThH\psi(l)}{2a\cdot(1-\nu_{xz}\nu_{zx})} = 0 \tag{21}$$

$$D_2\psi_{2,x} + D_3\psi_{3,x} - D_4\psi_{4,x} - \frac{E_x ThH\psi(l)}{2a\cdot(1-\nu_{xz}\nu_{zx})} = 0 \tag{22}$$

At this stage, DQM can be applied to the system of the differential equilibrium equations and their boundary conditions, forming

$$\frac{s}{4\pi^2}\left(\frac{t_k}{T}\right)^2\left(\frac{L}{l_k}\right)^2 \sum_{j=1}^{N_k} C_{ijk}^{(2)}\Psi_{jk} - \delta_{ij}\Psi_{jk} - \frac{1}{l_k}\sum_{j=1}^{N_k} C_{ijk}^{(1)} W_{jk} = 0$$
$$i = 1,\dots,N, \quad k = 1,2,3,4 \tag{23}$$

$$\sum_{j=1}^{N_k} C_{ijk}^{(1)}\Psi_{jk} + \frac{1}{l_k}\sum_{j=1}^{N_k} C_{ijk}^{(2)} W_{jk} = \lambda\cdot\left(\frac{s}{l_k}\right)\sum_{j=1}^{N_k} C_{ijk}^{(2)} W_{jk}$$
$$i = 1,\dots,N, \quad k = 1,2,3,4 \tag{24}$$

where $C_{ijk}^{(1)}$ and $C_{ijk}^{(2)}$ are the weighting coefficients for the first and second derivatives along the nondimensional X axis, respectively; W_{jk} and Ψ_{jk} are the deflection and rotation of the jth point in the kth section, respectively; δ_{ij} is the Kronecker delta; and s is the shear deformation parameter defined by

$$s = \frac{4\pi^2 D_1}{kGTL^2} \tag{25}$$

which represents the effect of shear deformation. Define λ as

$$\lambda = \frac{P}{4\pi^2 D_1/L^2} \tag{26}$$

The boundary conditions of a beam plate with its far edges clamped can be represented by

$$W_{11} = \Psi_{11} = 0, \qquad W_{N_4 4} = \Psi_{N_4 4} = 0$$
$$W_{N_1 1} = W_{12} = W_{13} \tag{27a}$$

$$\Psi_{N_1 1} = \Psi_{12} = \Psi_{13}, \qquad W_{N_2 2} = W_{N_3 3} = W_{14}$$
$$\Psi_{N_2 2} = \Psi_{N_3 3} = \Psi_{14} \tag{27b}$$

$$\frac{t_1^3}{l_1}\sum_{j=1}^{N_1} C_{N_1 j1}^{(1)}\Psi_{j1} - \frac{t_2^3}{l_2}\sum_{j=1}^{N_2} C_{1j2}^{(1)}\Psi_{j2} - \frac{t_3^3}{l_3}\sum_{j=1}^{N_3} C_{1j3}^{(1)}\Psi_{j3}$$
$$+ \frac{6t_1\cdot t_2\cdot t_3}{l_2}\Psi_{N_1 1} = 0 \tag{27c}$$

$$\frac{t_4^3}{l_4}\sum_{j=1}^{N_4} C_{1j4}^{(1)}\Psi_{j4} - \frac{t_2^3}{l_2}\sum_{j=1}^{N_2} C_{N_2 j2}^{(1)}\Psi_{j2} - \frac{t_3^3}{l_3}\sum_{j=1}^{N_3} C_{N_3 j3}^{(1)}\Psi_{j3}$$
$$- \frac{6t_1\cdot t_2\cdot t_3}{l_2}\Psi_{14} = 0 \tag{27d}$$

$$\frac{t_1}{l_1}\sum_{j=1}^{N_1} C_{N_1 j1}^{(1)} W_{j1} - \frac{t_2}{l_2}\sum_{j=1}^{N_2} C_{1j2}^{(1)} W_{j2} - \frac{t_3}{l_3}\sum_{j=1}^{N_3} C_{1j3}^{(1)} W_{j3} = 0 \tag{27e}$$

$$\frac{t_4}{l_4}\sum_{j=1}^{N_4} C_{1j4}^{(1)} W_{j4} - \frac{t_2}{l_2}\sum_{j=1}^{N_2} C_{N_2 j2}^{(1)} W_{j2} - \frac{t_3}{l_3}\sum_{j=1}^{N_3} C_{N_3 j3}^{(1)} W_{j3} = 0 \tag{27f}$$

where t_k and l_k, $k = 1, 2, 3, 4$, represent the thickness and length of each section. The imposition of these boundary conditions makes some of the terms of Eqs. (23) and (24) redundant. To eliminate this redundancy, the terms corresponding to $i = 1$ and N_k in Eqs. (23) and (24) for all regions have to be omitted. The combination of Eqs. (23), (24), and the preceding 16 boundary conditions produces the following system of linear equations:

$$\begin{bmatrix} A_{bb} & A_{bi} \\ A_{ib} & A_{ii} \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} \Psi_b \\ W_b \end{Bmatrix} \\ \begin{Bmatrix} \Psi_i \\ W_i \end{Bmatrix} \end{Bmatrix} = \lambda\cdot \begin{bmatrix} 0 & 0 \\ B_{ib} & B_{ii} \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} \Psi_b \\ W_b \end{Bmatrix} \\ \begin{Bmatrix} \Psi_i \\ W_i \end{Bmatrix} \end{Bmatrix} \tag{28}$$

where the subscripts b and i are boundary and interior points used for writing the differential quadrature, respectively. The vectors $\{\Psi\}$ and $\{W\}$ contain the rotations and normal deflections corresponding to the boundary and interior points. Transforming Eq. (28) into a general eigenvalue form in terms of $\{W_i\}$ results in

$$[A^*]\cdot\{W_i\} = \lambda[B^*]\cdot\{W_i\} \tag{29}$$

The solution of the preceding eigenvalue problem by a standard eigensolver provides the eigenvalues, which are the buckling loads, and the eigenvectors $\{W_i\}$, which are the corresponding buckling mode shapes.

Results and Discussion

The foregoing formulation was implemented in a computer program. The program was used to investigate the influence of several parameters on the buckling response of laminated beam plates. It was assumed that the beam plates' outermost edges were clamped.

Table 1 Normalized buckling loads ($\bar{P}_{cr}$) for a beam plate with single delamination, $h/T = 0.5$ ($s = 0$)[a]

			Lee et al.[6]		DQM (present study)	
a/L	Simitses[12]	Chen[3]	Symmetric	Anti-symmetric	Symmetric	Asymmetric
0.1	0.9999	0.9999	0.9999	1.9480	0.9999	0.9999
0.2	0.9956	0.9956	0.9956	1.4360	0.9956	0.9956
0.3	0.9638	0.9638	0.9639	1.0240	0.9638	0.9638
0.4	**0.8481**	**0.8561**	**0.8562**	**0.8482**	**0.8561**	**0.8481**
0.5	0.6896	0.6896	0.6898	0.7967	0.6896	0.6896
0.6	0.5411	0.5411	0.5413	0.7929	0.5411	0.5411
0.7	0.4310	0.4310	0.4311	0.7629	0.4310	0.4310
0.8	0.3514	0.3514	0.3515	0.6857	0.3514	0.3514
0.9	0.2923	0.2933	0.2934	0.5947	0.2933	0.2933

[a]Boldface indicates that the antisymmetric mode at $a/L = 0.4$ is the first buckling mode for this value of a/L.

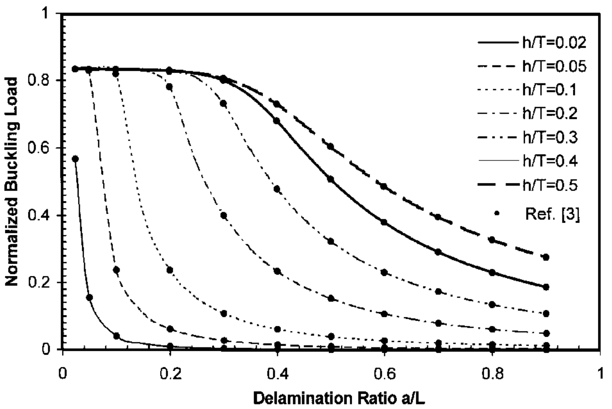

Fig. 2 Buckling loads for different delamination configurations (s = 0.2).

To verify the accuracy and reliability of the proposed methodology, DQM is used to analyze several delamination buckling problems, and the results are compared with those reported by other researchers. Table 1 tabulates the buckling loads obtained by DQM for a specially orthotropic composite laminate containing a delamination at its midplane, i.e., $h/T = 0.5$. This problem is analyzed for various delamination-to-length ratios a/L. In these analyses, the effect of shear deformation is neglected. The buckling loads are normalized with respect to the Euler buckling load of a perfect column, that is,

$$\bar{P}_{cr} = \frac{P_{cr}}{4\pi^2 D_1/L^2} \tag{30}$$

The symmetric and antisymmetric notations in Table 1 conform to the terminology used by Lee et al.[6] Note that the results reported by Chen[3] were obtained by imposing the symmetry condition in the axial direction, whereas Simitses et al.[12] did not apply such an imposition. This is an important consideration as the antisymmetric buckling mode can occurs at specific a/L ratios (see Fig. 6 of Ref. 6); therefore, by imposing the symmetry condition, accurate simulation of the buckling modes for all a/L ratios is not achievable. The result reported by Lee et al.[6] are provided for the two possible conditions. Also, note that the nonsymmetric terminology used to identify some of the DQM results conforms to the terminology used by Simitses et al.[12] As can be seen, there is an excellent agreement between the results obtained by DQM and those of Refs. 3, 6, and 12. Indeed, the DQM results are closer to the results obtained based on the analytical solutions[3,12] than those calculated based on the finite element analysis of Lee et al.[6]

One of the parameters investigated was the influence of the shear deformation. For this, the resulting buckling response of beam plates having a symmetric delamination with various configurations, with a shear deformation factor $s = 0.2$, is shown in Fig. 2. The variables considered were the length and the through-the-thickness location of the delamination. As can be seen, the DQM results are in excellent agreement with those obtained from the analytical solution

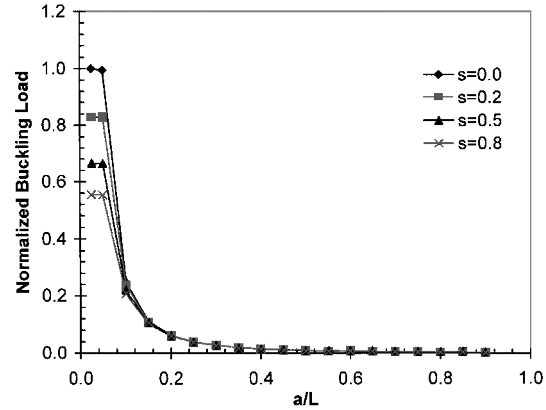

Fig. 3 Effect of shear deformation on the buckling strength of beam plates with $h/T = 0.05$.

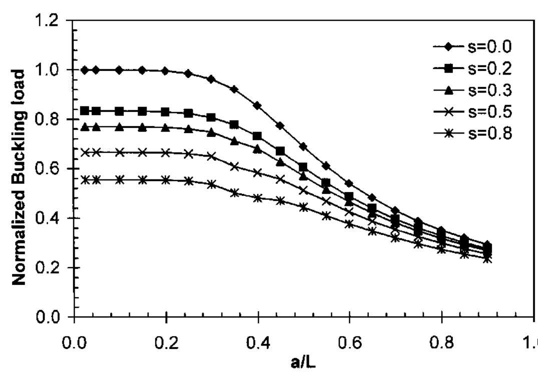

Fig. 4 Effect of shear deformation on the buckling strength of beam plates with $h/T = 0.5$.

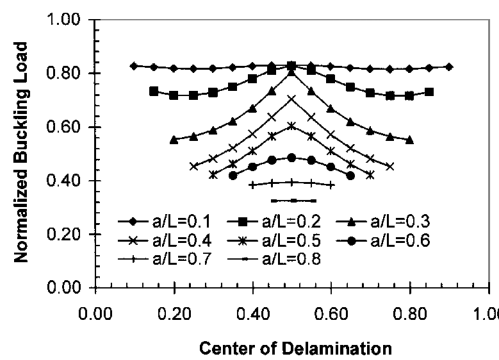

Fig. 5 Effect of the delamination position on the buckling strength of beam plates with $h/T = 0.5$.

of Chen,[3] to the extent that the differences are indistinguishable. Moreover, as shown, the classical laminate theory generally overestimates the buckling response of the composites, especially those with short delamination length. This phenomenon is more noticeable in Figs. 3 and 4, where the buckling strength of beam plates, with $h/T = 0.05$ and 0.5 and various delamination lengths, is calculated for a practical range of shear deformation factors. Figures 3 and 4 show that the buckling strength of composites having thicker sublaminates ($h/T = 0.5$) decreases as their shear deformation increases. This behavior is quite consistent regardless of the length of delamination, yet it is more significant for $a/L < 0.6$. On the other hand, when the sublaminate is thin, the shear effect becomes noticeable only at very small delamination lengths, e.g., $a/L < 0.1$.

The influence of the longitudinal delamination position on the buckling response of the plates is shown in Fig. 5 for beam plates with $h/T = 0.5$. For the delamination length ratios $0.2 < a/L < 0.6$, the buckling response varies depending on where the delamination is situated along the span of the beams. As expected, the highest buckling strength is obtained when the delamination is at the center of the span of the beam. The behavior is consistent throughout, including in beams with h/T ratios other than 0.5, as shown in Fig. 5.

Conclusions

The buckling response of a one-dimensional beam plate having an across-the-width delamination that is located in an arbitrary location (through the thickness and along the length of the beam plate) was characterized by the DQM. A beam theory with shear deformation was used to formulate the problem. Using DQM, the system of the differential equations was transformed into a system of linear algebraic eigenvalue equations, solvable by any standard eigensolver.

Based on the foregoing analysis, the following conclusions are made:

1) The delamination buckling strength is very sensitive to the shear deformation. The buckling strengths are generally lower than those predicted by the classical laminate theory.

2) The buckling strength of beams with delamination length ratios $0.2 < a/L < 0.6$ is significantly influenced by the eccentricity of the delaminations (eccentricity with respect to the midplane).

3) The accuracy of the results is governed by the proper selection of weighting functions and the sampling points.

In summary, the simplicity, efficiency, and accuracy of DQM suggest that the method can be considered as a suitable tool for characterizing the delamination buckling response of composites.

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